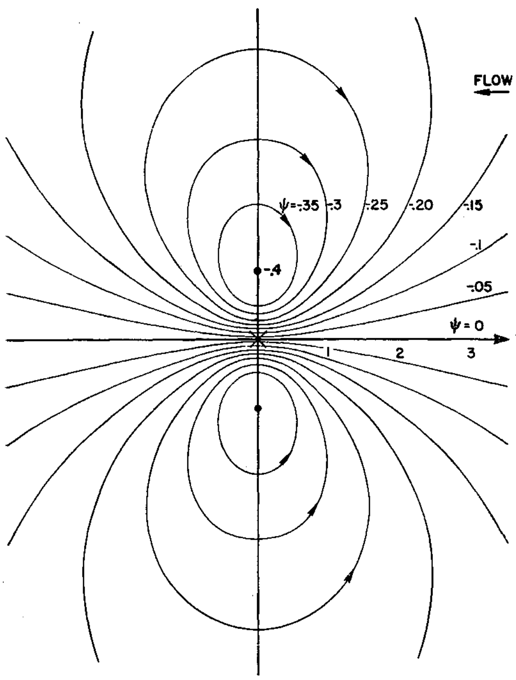

Fig. 3 Illustration of the current pattern induced in the plane of the flow by the Hall effect

has zero divergence, it may be written

$$\mathbf{j}/N_e eU = (\partial\psi/\bar{r}\partial\theta, \quad -\partial\psi/\partial\bar{r}, \quad j_z/N_e eU) \tag{6}$$

Furthermore, the electric field vector is irrotational and lies in the plane of the flow. Thus, it may be derived from a potential $\phi(\bar{r}, \theta)$:

$$\mathbf{E}e\tau/Um = (\partial\phi/\partial\bar{r}, \quad \partial\phi/\bar{r}\partial\theta, \quad 0) \tag{7}$$

Substituting these expressions for **j**, **E**, **v**, and **B** into the Ohm's law, one finds

$$\begin{aligned} \partial\psi/\partial\theta &= -\bar{r}\partial\phi/\partial\bar{r} + j_z/N_e eU \\ \partial\psi/\partial\bar{r} &= -\partial\phi/\bar{r}\partial\theta \\ j_z/N_e eU &= -(1/\bar{r})[1 - (\bar{a}^2/\bar{r}^2)]\cos\theta - (1/\bar{r}^2)(\partial\psi/\partial\theta) \end{aligned} \tag{8}$$

From these equations it is clear that one can find a solution of the form

$$\begin{aligned} \psi(\bar{r}, \theta) &= G(\bar{r})\sin\theta \\ \phi(\bar{r}, \theta) &= F(\bar{r})\cos\theta \\ j_z/N_e eU &= J(\bar{r})\cos\theta \end{aligned} \tag{9}$$

The equation for G which results is

$$\bar{r}(d/d\bar{r})[\bar{r}(dG/d\bar{r})] - [1 + (1/\bar{r}^2)]G = (1/\bar{r})[1 - (\bar{a}^2/\bar{r}^2)] \tag{10}$$

The general solution of this equation which tends to zero as $\bar{r} \to \infty$ is

$$G(\bar{r}) = AI_1(1/\bar{r}) + K_1(1/\bar{r}) - \bar{r} + \bar{a}^2/\bar{r} \tag{11}$$

where A is an arbitrary constant and I_p and K_p are the modified Bessel functions of the first and second kinds of order p as defined, say, by Watson.[2] The corresponding solutions for $F(\bar{r})$ and $J(\bar{r})$ are

$$\begin{aligned} F(\bar{r}) &= [-AI_1'(1/\bar{r}) - K_1'(1/\bar{r}) - \bar{r}^2 - \bar{a}^2]/\bar{r} \\ J(\bar{r}) &= [-AI_1(1/\bar{r}) - K_1(1/\bar{r})]/\bar{r}^2 \end{aligned} \tag{12}$$

The boundary condition that determines A normally would be applied on the surface of the cylinder $\bar{r} = \bar{a}$. If the cylinder is a conductor, it is an equipotential and $F(\bar{a}) = 0$. If it is an insulator, the radial component of the current vanishes, so that $G(\bar{a}) = 0$. For the sake of brevity, only the case $\bar{a} = 0$ corresponding to a uniform unperturbed flow will be considered. The appropriate boundary condition at $\bar{r} = 0$ is then $G(0)$ finite, so that $A = 0$. The current in the z direction then is given by

$$j_z/N_e eU = -(\omega\tau)^2 K_1(\omega\tau)\cos\theta \tag{13}$$

With no cylinder present, the current in the absence of Hall effect reduces from Eq. (4) to

$$j_z/N_e eU = -\omega\tau\cos\theta \tag{14}$$

Elementary discussions of the Hall effect usually state that the conductivity normal to the field lines is reduced by the factor $[1 + (\omega\tau)^2]^{-1}$. This result ignores the electric field induced by the Hall effect which depends on the geometry, but if it is applied to this case, one finds

$$j_z/N_e eU = -\omega\tau[1 + (\omega\tau)^2]^{-1}\cos\theta \tag{15}$$

Equations (13–15) are plotted in Fig. 2. It is seen that j_z is reduced substantially by the Hall effect when $\omega\tau$ is of the order of unity, and that Eq. (15) gives a reasonably accurate estimate of j_z except where $\omega\tau$ is large. In this region the induced electric field cannot be ignored. The current lines in the plane are given by

$$\psi = [K_1(\omega\tau) - (\omega\tau)^{-1}]\sin\theta \tag{16}$$

and these are shown in Fig. 3. The equipotentials are given by

$$\phi = [\omega\tau K_0(\omega\tau) + K_1(\omega\tau) - (\omega\tau)^{-1}]\cos\theta \tag{17}$$

Note that the cylinder defined by $\omega\tau \approx 1.1$ is an equipotential.

References

[1] Spitzer, L., Jr., *Physics of Fully Ionized Gases* (Interscience Publishers Inc., New York, 1956).

[2] Watson, G. N., *Bessel's Functions* (Macmillan Co., New York, 1944).

Propagation of Thermal Disturbances in Rarefied Gas Flows

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Grad and Ai recently have shown that, when the flow time is much less than the average collision time or relaxation time in a rarefied gas field, the one-dimensional linearized Grad equations indicate the propagation of disturbances along two distinct characteristics.[1, 2] The phenomena of propagation of signals along two distinct wave fronts in the rarefied gas field are strikingly similar to the propagation phenomena observed for waves of finite amplitude in continuum flows.[3]

The propagation phenomena in the rarefied gas field occur only in the limit of $t/t_f \ll 1$, where t_f is the relaxation time. However, in extremely rarefied gases ($<10^{10}$ particles per unit volume), the disturbance propagation can occur over very large distances. The similarity of the behavior in this limit

Received by IAS October 26, 1962; revision received November 6, 1962.

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with one-dimensional nonsteady wave propagation phenomena in continuum flows suggests that solutions of the one-dimensional Grad equations for small disturbances[1, 2]

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau}{\partial x} = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} + \frac{5}{3}\frac{\partial u}{\partial x} + \frac{2}{3p_0}\frac{\partial q}{\partial x} = 0 \tag{3}$$

$$\frac{\partial \tau}{\partial t} + \frac{8}{15}\frac{\partial q}{\partial x} + \frac{4}{3}p_0\frac{\partial u}{\partial x} = -\frac{p_0}{\mu_0}\tau \tag{4}$$

$$\frac{\partial q}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial \tau}{\partial x} + \frac{5}{2}\frac{p_0{}^2}{\rho_0}\frac{\partial \theta}{\partial x} = -\frac{p_0}{\mu_0}q \tag{5}$$

$$p = s + \theta \tag{6}$$

can be obtained in the form

$$[\partial/\partial t \pm c(\partial/\partial x)][\alpha p + \beta\theta + \gamma\tau + \delta q + \epsilon u] = 0 \tag{7}$$

Here p is the perturbation pressure, s the perturbation density, θ the perturbation temperature, u the velocity, τ the normal stress, q the heat flux, c the characteristic propagation velocity, and α, β, γ, δ, ϵ are constants. A relaxation time also can be defined by $t_f \simeq \mu_0/p_0$.

When the relaxation time is much greater than the flow time, $t_f \gg t$, Eqs. (4) and (5) can be written in the approximate form

$$\frac{\partial \tau}{\partial t} + \frac{8}{15}\frac{\partial q}{\partial x} + \frac{4}{3}p_0\frac{\partial u}{\partial x} = 0 \tag{4'}$$

$$\frac{\partial q}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial \tau}{\partial x} + \frac{5}{2}\frac{p_0{}^2}{\rho_0}\frac{\partial \theta}{\partial x} = 0 \tag{5'}$$

Since solutions are sought in the form (7), the equations can be rewritten as

$$\frac{\partial p}{\partial t} - \frac{\partial \theta}{\partial t} + c_1\frac{\partial}{\partial x}\frac{u}{c_1} = 0 \tag{8}$$

$$\frac{\partial}{\partial t}\left(\frac{c_1}{c_0}\right)^2\frac{u}{c_1} + c_1\frac{\partial p}{\partial x} + c_1\frac{\partial}{\partial x}\frac{\tau}{p_0} = 0 \tag{9}$$

$$\frac{\partial p}{\partial t} + c_1\frac{\partial}{\partial x}\left(\frac{5}{3}\frac{u}{c_1}\right) + c_1\frac{\partial}{\partial x}\left(\frac{2}{3}\frac{q}{p_0c_1}\right) = 0 \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{4}\frac{\tau}{p_0}\right) + c_1\frac{\partial}{\partial x}\left(\frac{2}{5}\frac{q}{p_0c_1}\right) + c_1\frac{\partial}{\partial x}\frac{u}{c_1} = 0 \tag{11}$$

$$\frac{\partial}{\partial t}\left(\frac{c_1}{c_0}\right)^2\frac{q}{p_0c_1} + c_1\frac{\partial}{\partial x}\frac{\tau}{p_0} + c_1\frac{\partial}{\partial x}\frac{5}{2}\theta = 0 \tag{12}$$

with

$$c_0{}^2 = p_0/\rho_0$$

On multiplying Eqs. (9–12) by the constants α, β, γ, δ, respectively, the following equations must be satisfied:

$$1 + \beta = \alpha$$

$$-1 = \tfrac{5}{2}\delta$$

$$\alpha(c_1/c_0)^2 = 1 + \tfrac{5}{3}\beta + \gamma$$

$$\tfrac{3}{4}\gamma = \alpha + \delta$$

$$\delta(c_1/c_0)^2 = \tfrac{2}{3}\beta + \tfrac{2}{5}\gamma$$

Two solutions result with propagation velocities identical with the values obtained by Grad[1] and Ai,[2] $c_1/c_0 = (0.661)^{1/2}$ and $c_1/c_0 = (4.54)^{1/2}$.

The resulting characteristic equations are

$$[(\partial/\partial t) \pm 0.813c_0(\partial/\partial x)]P_{1\pm} = 0 \tag{13}$$

$$[(\partial/\partial t) \pm 2.13c_0(\partial/\partial x)]P_{2\pm} = 0 \tag{14}$$

where

$$P_{1\pm} = [\theta - 0.51p - 0.11(\tau/p_0)] \pm (1/c_0)[0.33(q/p_0) - 0.42u]$$

$$P_{2\pm} = [\theta + 0.78p + 1.18(\tau/p_0)] \pm (1/c_0)[0.85(q/p_0) + 1.66u]$$

Assuming the existence of external heat addition $H(x, t)$ and external forces $F(x, t)^2$ and including changes in the characteristic quantities as $t \to t_f$, the equations can be written in a nondimensional form

$$\left(\frac{\partial}{\partial t} \pm 0.813\frac{\partial}{\partial x}\right)P_{1\pm} = \left(0.487\frac{H}{p_0} \mp 0.417\frac{F}{c_0}\right)\frac{L}{c_0} + \frac{L}{t_fc_0}\left(0.15\frac{\tau}{p_0} \mp 0.4\frac{q}{p_0c_0}\right) \tag{13'}$$

$$\left(\frac{\partial}{\partial t} \pm 2.13\frac{\partial}{\partial x}\right)P_{2\pm} = \left(1.78\frac{H}{p_0} \pm 1.66\frac{F}{c_0}\right)\frac{L}{c_0} - \frac{L}{t_fc_0}\left(\frac{1.57\tau}{p_0} \pm 0.4\frac{q}{p_0c_0}\right) \tag{14'}$$

on multiplying both sides of the equation by L/c_0. In Eqs. (13′) and (14′), $(c_0/x')(L/c_0)$ is replaced by $1/x$, and $(1/t') \times (L/c_0)$ is replaced by $1/t$. The validity of the characteristic approximation now can be determined from the condition $L/t_fc_0 \ll 1$, where L can be related to the distance over which the propagation occurs. The terms multiplying L/t_fc_0 provide a correction when the flow time L/c_0 approaches the relaxation time.

References

[1] Grad, H., "On the kinetic theory of rarefied gases," Communs. Pure and Appl. Math. **2**, 331–407 (December 1949).

[2] Ai, D. K., "Small perturbations in the unsteady flow of a rarefied gas based on Grad's thirteen moment approximation," Calif. Inst Tech., GALCIT Rept. 59 (September 20, 1960).

[3] Rudinger, G., *Wave Diagrams for Nonsteady Flow in Ducts* (D. Van Nostrand Co., Inc.. Princeton, N. J., 1955), pp. 15–19.

Isothermal Compressibilities of Two Liquid Monopropellants

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LIQUID monopropellants find wide use as gas generants for powering propellant pump turbines and as auxiliary power supplies in ballistic missiles. When the adiabatic

Received by ARS August 7, 1962; revision received December 18, 1962. The work reported here has been submitted by the author to Drexel Institute of Technology in partial fulfillment of the requirements for the M.S. degree. J. Thygeson, of the Drexel Institute of Technology, made several constructive criticisms throughout this work.

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